# On Curves over Finite Fields with Jacobians of Small Exponent

KEVIN FORD
Department of Mathematics, 1409 West Green Street
University of Illinois at Urbana-Champaign
Urbana, IL 61801, USA
ford@math.uiuc.edu

IGOR SHPARLINSKI
Department of Computing, Macquarie University
Sydney, NSW 2109, Australia
igor@ics.mq.edu.au

September 21, 2008

**Abstract**

We show that finite fields over which there is a curve of a given genus $g \geq 1$ with its Jacobian having a small exponent, are very rare. This extends a recent result of W. Duke in the case $g = 1$. We also show when $g = 1$ or $g = 2$, our lower bounds on the exponent, valid for almost all finite fields $\mathbb{F}_q$ and all curves over $\mathbb{F}_q$, are best possible.

**Keywords:** Jacobian, group structure, distribution of divisors

**2000 Mathematics Subject Classification:** 11G20, 11N25, 14H40

## 1 Introduction

Let $\mathcal{J}_\mathcal{C}(\mathbb{F}_q)$ denote the Jacobian of a curve $\mathcal{C}$ defined over a finite field $\mathbb{F}_q$ of $q$ elements. We denote by $\ell_q(C)$ the exponent of $\mathcal{J}_\mathcal{C}(\mathbb{F}_q)$ (that is, $\ell_q(\mathcal{C})$ is the

largest order of elements of the Abelian group $\mathcal{J}_\mathcal{C}(\mathbb{F}_q)$) and by $g$ the genus of $\mathcal{C}$. We start with recalling two well know facts.

- The Weil bound implies that

$$(q^{1/2}-1)^{2g} \le \#\mathcal{J}_\mathcal{C}(\mathbb{F}_q) \le (q^{1/2}+1)^{2g}, \qquad (1)$$

  see Corollary 5.70, Theorem 5.76 and Corollary 5.80 of [1]. In particular, for fixed $g$,

$$\#\mathcal{J}_\mathcal{C}(\mathbb{F}_q) = q^g + O_g(q^{g-1/2}).$$

- The Jacobian $\mathcal{J}_\mathcal{C}(\mathbb{F}_q)$ is an Abelian group with at most $2g$ generators, that is, for some positive integers $m_1, \ldots, m_{2g}$ we have

$$\mathcal{J}_\mathcal{C}(\mathbb{F}_q) \cong \mathbb{Z}/m_1\mathbb{Z} \times \ldots \times \mathbb{Z}/m_{2g}\mathbb{Z}, \quad \text{where} \quad m_1 \mid \ldots \mid m_{2g}, \qquad (2)$$

  (in particular $m_1 = \ldots = m_j = 1$ if the rank of $\mathcal{J}_\mathcal{C}(\mathbb{F}_q)$ is $2g-j$) and also

$$m_i | (q-1) \qquad (1 \le i \le g), \qquad (3)$$

  see Proposition 5.78 of [1].

Thus we see $\ell_q(\mathcal{C}) = m_{2g}$ where $m_{2g}$ is defined by the representation (2), which together with (1) implies the following trivial bound

$$\ell_q(\mathcal{C}) \ge (\#\mathcal{J}_\mathcal{C}(\mathbb{F}_q))^{1/2g} \ge q^{1/2}-1. \qquad (4)$$

For elliptic curves $\mathcal{C} = \mathcal{E}$ over finite fields the exponent $\ell_q(\mathcal{E})$ has been studied in a number of works, see [3, 8, 9, 13, 14], with a variety of results, each of them indicating that in a "typical case" $\ell_q(\mathcal{E})$ tends to be substantially larger than the bound (4) guarantees. However for general curves the behavior of $\ell_q(\mathcal{C})$ has not been studied. Let $\pi(x)$ denote the number of primes $p \le x$. W. Duke [3, footnote on page 691], among other results, has proved that for a sufficiently large $x$ and all but $o(\pi(x))$ of prime powers $q \le x$, the bound

$$\ell_q(\mathcal{E}) \ge q^{3/4}/\log q \qquad (5)$$

holds for all elliptic curves $\mathcal{E}$ defined over $\mathbb{F}_q$ (the paper [3] considers only primes, but including all prime powers in the statement is trivial of course).

We provide a generalization and some improvement of (5) for curves of arbitrary genus.

**Theorem 1.** *Fix $g \geq 1$ and let $\varepsilon(x)$ be a positive, decreasing function of $x$ with $\varepsilon(x) \to 0$ as $x \to \infty$. For all but $o(\pi(x))$ of the prime powers $q \leq x$, the bound*

$$\ell_q(\mathcal{C}) \geq q^{3/4+\varepsilon(q)}$$

*holds for all curves $\mathcal{C}$ of genus $g$ defined over $\mathbb{F}_q$.*

The method of proof of (5), used in [3], is somewhat specific to elliptic curves, so here we use a slightly different approach to counting fields $\mathbb{F}_q$ that may contain a "bad" curve.

We show that Theorem 1 is best possible for $g = 1$ and $g = 2$. In particular, the bound (5) of W. Duke [3] is quite sharp.

**Theorem 2.** *For any fixed $\varepsilon > 0$ there exists $\alpha > 0$ such that for sufficiently large $x$, there are at least $\alpha\pi(x)$ primes $q \leq x$ such that for some nonsupersingular elliptic curve $\mathcal{E}$ and some nonsupersingular curve $\mathcal{C}$ of genus $g = 2$ defined over $\mathbb{F}_q$, the bounds*

$$\ell_q(\mathcal{E}) \leq q^{3/4+\varepsilon} \qquad \text{and} \qquad \ell_q(\mathcal{C}) \leq q^{3/4+\varepsilon}$$

*hold.*

The proof is based on a special case of a certain lower bound on the number of shifted primes $p - 1$ having a divisor in a given interval. In full generality this bound is given in Theorem 7 of [5]. Such results have been applied to study the order of a given integer $a > 1$ modulo almost all primes $p$, see [4, 7, 10], and now they have turned out to be useful for studying exponents of Jacobians. This argument also immediately implies the following result which applies to all curves over $\mathbb{F}_q$ of all possible genera.

**Theorem 3.** *Let $\varepsilon(x)$ be a positive, decreasing function of $x$ with $\varepsilon(x) \to 0$ as $x \to \infty$. For all but $o(\pi(x))$ of the prime powers $q \leq x$, the bound*

$$\ell_q(\mathcal{C}) \geq q^{1/2+\varepsilon(q)}$$

*holds for all curves $\mathcal{C}$ of arbitrary genus defined over $\mathbb{F}_q$.*

Throughout the paper, the implied constants in the symbols '$O$', '$\ll$' and '$\gg$' do not depend on any parameter unless indicated by a subscript, that is, $O_g$, $\ll_g$ or $\gg_g$ (we recall that the notations $U = O(V)$, $U \ll V$, and $V \gg U$ are all equivalent to the assertion that the inequality $|U| \leq cV$ holds for some constant $c > 0$).

**Acknowledgments.** The authors would like to thank Bjorn Poonen for a number of very helpful discussions. During the preparation of this paper, K. F. was supported by NSF grants DMS-0301083 and DMS-0555367 and I. S. was supported in part by ARC grant DP0556431.

## 2 Preliminaries

We have already mentioned that our results are based on some estimates from [5] on shifted primes having a divisor in a given interval. Here we give a brief guide to these estimates.

As in [5] we use $H(x, y, z)$ to denote the number of positive integers $n \le x$ having a divisor $d$ with $y < d \le z$. Theorem 1 of [5] gives the right order of magnitude of $H(x, y, z)$ in the full range of parameters. However for our purposes we need only the estimate

$$H(x, y, z) \ll xu^{\delta}(\log(2/u))^{-3/2} \tag{6}$$

where

$$\delta = 1 - \frac{1 + \log\log 2}{\log 2} = 0.086071\ldots$$

and $u$ is defined by the equation $y^{1+u} = z$, which holds uniformly in the range $2y \le z \le y^2$, $3 \le y \le \sqrt{x}$.

Furthermore, we need the upper bound on $H(x, y, z)$ only as tool of estimating $H(x, y, z, \mathcal{P}_\lambda)$ which is the number of primes $p \le x$ such that $p + \lambda$ has a divisor $d$ with $y < d \le z$. Theorem 6 of [5] gives the upper bound

$$H(x, y, z, \mathcal{P}_\lambda) \ll \frac{H(x, y, z)}{\log x} \tag{7}$$

which holds for every fixed non-zero integer $\lambda$ in the range $z \ge y + (\log y)^{2/3}$ and $3 \le y \le \sqrt{x}$, which is much wider than is necessary for the purposes of this paper.

We also need Theorem 7 of [5] which gives a lower bound on $H(x, y, z, \mathcal{P}_\lambda)$ in a certain range of $x, y, z$. However, since its proof is quite short, we give an independent derivation in Section 4.

## 3 Proof of Theorem 1

The number of prime powers $q = p^a \le x$ with $a \ge 2$ is $O(x^{1/2})$. Thus, it suffices to show that for all but $o(x/\log x)$ of the primes $q$ with $x/2 < q \le x$, the bound

$$\ell_q(\mathcal{C}) \ge q^{3/4+\varepsilon(q)}$$

holds for all curves $\mathcal{C}$ of genus $g$ defined over $\mathbb{F}_q$.

For a $(2g-1)-$tuple $\mathbf{k} = (k_1, \ldots, k_{2g-1})$ of positive integers, we consider the set $\mathcal{Q}_\mathbf{k}$ of primes $x/2 \le q \le x$ for which there exists a curve $\mathcal{C}$ of genus $g \ge 1$ over $\mathbb{F}_q$ such that $m_1 = k_1$, $m_i = m_{i-1}k_i$, where $m_i$ is as in (2) and (3), $i = 1, \ldots, 2g-1$. In particular, if such a curve $\mathcal{C}$ exists, then

$$q - 1 \equiv 0 \pmod{k_1 \ldots k_g}. \tag{8}$$

Since

$$k_1^{2g} k_2^{2g-1} \ldots k_{2g-1}^2 | \#\mathcal{J}_\mathcal{C}(\mathbb{F}_q),$$

we see by (1) that there are at most

$$U_\mathbf{k} = \frac{(x^{1/2}+1)^{2g}}{k_1^{2g} k_2^{2g-1} \ldots k_{2g-1}^2} \tag{9}$$

possibilities for the cardinality $N = \#\mathcal{J}_\mathcal{C}(\mathbb{F}_q)$.

For each of such values $N$, we see by (1) that

$$N^{1/g} - 2N^{1/2g} + 1 \le q \le N^{1/g} + 2N^{1/2g} + 1.$$

Recalling (8) we deduce that for each possible cardinality $N$ the prime powers $q$ may take at most

$$V_\mathbf{k} = \frac{5(x^{1/2}+1)}{k_1 k_2 \ldots k_g} + 1 \tag{10}$$

values. Therefore, combining (9) and (10), we derive

$$\#\mathcal{Q}_\mathbf{k} \le U_\mathbf{k} V_\mathbf{k} \le \frac{5(x^{1/2}+1)^{2g+1}}{k_1^{2g+1} k_2^{2g} \ldots k_g^{g+2} k_{g+1}^{g} \ldots k_{2g-1}^2} + \frac{(x^{1/2}+1)^{2g}}{k_1^{2g} k_2^{2g-1} \ldots k_{2g-1}^2}. \tag{11}$$

When $g = 1$, we interpret the right side as $5(x^{1/2}+1)^3 k_1^{-3} + (x^{1/2}+1)^2 k_1^{-2}$.

For any curve $\mathcal{C}$ of genus $g \geq 1$ over $\mathbb{F}_q$ and any positive integer $s \leq 2g-1$, we have

$$\ell_q(\mathcal{C}) = m_{2g} \geq \left(\frac{\#\mathcal{J}_{\mathcal{C}}(\mathbb{F}_q)}{m_1 \ldots m_s}\right)^{1/(2g-s)} \geq \left(\frac{(q^{1/2}-1)^{2g}}{k_1^s k_2^{s-1} \ldots k_s}\right)^{1/(2g-s)}. \qquad (12)$$

In fact, we only need (12) for $s = g$ and $s = 2g-1$.

Suppose without loss of generality that $\varepsilon(x) \geq (\log x)^{-1/2}$ and write $\eta = \varepsilon(x/2)$. Assume $x$ is large, in particular so large that

$$\eta < \frac{1}{100g}.$$

Let $I$ be the interval $(x^{1/4-3\eta}, x^{1/4+3\eta}]$. Let $\mathcal{K}$ denote the set of $\mathbf{k}$ satisfying

$$k_1 \cdots k_g \notin I, \qquad (13)$$

$$k_1^g k_2^{g-1} \ldots k_g \geq x^{g/4-2g\eta}, \qquad (14)$$

$$k_1^{2g-1} k_2^{2g-2} \ldots k_{2g-1} \geq x^{g-3/4-2\eta}. \qquad (15)$$

Partition the primes $q \in (x/2, x]$ into three sets: $\mathcal{T}_1$ is the set of such primes for which $q-1$ has a divisor in $I$, $\mathcal{T}_2$ is the set of such primes lying in a set $Q_{\mathbf{k}}$ with $\mathbf{k} \in \mathcal{K}$, and $\mathcal{T}_3$ is the set of remaining primes. By Theorems 1 and 6 of [5], that is, by a combination of (6) and (7), we have

$$\#\mathcal{T}_1 \ll \frac{x}{\log x} \eta^\delta (\log 1/\eta)^{-3/2} \qquad (16)$$

Now consider $q \in \mathcal{T}_2$. By (14),

$$k_1 \cdots k_g \geq (k_1^g k_2^{g-1} \cdots k_g)^{1/g} \geq x^{1/4-2\eta},$$

hence $k_1 \cdots k_g > x^{1/4+3\eta}$ by (13). Combined with (11), (15), and the inequality $k_i \leq (x^{1/2}+1)^{2g}$ for each $i$, we obtain

$$\begin{aligned}
\#\mathcal{T}_2 &\leq \sum_{\mathbf{k} \in \mathcal{K}} \#\mathcal{Q}_{\mathbf{k}} \\
&\leq \left(\frac{5(x^{1/2}+1)^{2g+1}}{x^{g-1/2+\eta}} + \frac{(x^{1/2}+1)^{2g}}{x^{g-3/4-2\eta}}\right) \sum_{\mathbf{k} \in \mathcal{K}} \frac{1}{k_1 \cdots k_{2g-1}} \\
&\leq \left(\frac{5(x^{1/2}+1)^{2g+1}}{x^{g-1/2+\eta}} + \frac{(x^{1/2}+1)^{2g}}{x^{g-3/4-2\eta}}\right) (2g \log(x^{1/2}+1)+1)^{2g-1} \\
&\ll_g (\log x)^{2g-1} (x^{1-\eta} + x^{3/4+2\eta}) \\
&\ll_g x^{1-\eta/2}.
\end{aligned}$$

Together with (16), we see that all but $o(x/\log x)$ primes $q \in (x/2, x]$ lie in $\mathcal{T}_3$. For $q \in \mathcal{T}_3$, the condition (13) holds, thus either (14) is false or (15) is false. In either case, the bound (12) implies that $\ell_q(\mathcal{C}) \gg_g x^{3/4+2\eta}$, and hence for large $x$

$$\ell_q(\mathcal{C}) \ge q^{3/4+\varepsilon(q)}$$

for any curve $\mathcal{C}$ of genus $g$ defined over $\mathbb{F}_q$. $\square$

# 4 Proof of Theorem 2

We start with the case $g = 1$.

Without loss of generality we can assume that $\varepsilon < 1/20$. Put

$$y = x^{1/4-\varepsilon} \qquad \text{and} \qquad z = x^{1/4-\varepsilon/2}.$$

Since $y > x^{1/5}$, an integer $k \le x$ can have at most 4 prime factors $p$ with $y < p \le z$. Hence, the set $\mathcal{P}$ of primes $x/\log x \le q \le x$ such that $q-1$ has a prime divisor $p$ with $y < p \le z$, is of cardinally least

$$\#\mathcal{P} \ge \frac{1}{4} \sum_{\substack{y<p\le z \\ p \text{ prime}}} \pi(x; p, 1) + O\left(\frac{x}{(\log x)^2}\right),$$

where, as usual, $\pi(x; k, a)$ is the number of primes $q \le x$ with $q \equiv a \pmod k$.

By the Bombieri-Vinogradov theorem (see, for example, Section 28 of [2]),

$$\sum_{\substack{y<p\le z \\ p \text{ prime}}} \left| \pi(x; p, 1) - \frac{1}{p-1}\pi(x) \right| \ll \frac{x}{(\log x)^2}.$$

Therefore

$$\#\mathcal{P} \ge \frac{1}{4}\pi(x) \sum_{\substack{y<p\le z \\ p \text{ prime}}} \frac{1}{p-1} + O\left(\frac{x}{(\log x)^2}\right) = \frac{1}{4}\pi(x) \sum_{\substack{y<p\le z \\ p \text{ prime}}} \frac{1}{p} + O\left(\frac{x}{(\log x)^2}\right).$$

By the Mertens theorem (see Theorem 4.1 of Chapter 1 in [11]),

$$\sum_{\substack{y<p\le z \\ p \text{ prime}}} \frac{1}{p} = \log\log z - \log\log y + o(1) = \log \frac{1-2\varepsilon}{1-4\varepsilon} + o(1),$$

thus for large $x$ we have $\#\mathcal{P} \geq \alpha\pi(x)$ for a positive $\alpha$ depending on $\varepsilon$. This result is a special case of Theorem 7 of [5], but we include the proof because it is short.

For a sufficiently large $x$ and for any $q \in \mathcal{P}$, there are at least $2q^{1/2}z^{-2} - 1 \geq q^{\varepsilon}$ integers $k \in [q+1-2q^{1/2}, q]$ with $p^2|k$ for some prime $p|q-1$ with $y < p \leq z$. For any such $k$, by [12, 16, 17] one can always find an elliptic curve $\mathcal{E}$ over $\mathbb{F}_q$ with $\mathcal{E}(\mathbb{F}_q) = k$ of $\mathbb{F}_q$-rational points and the exponent $\ell_q(\mathcal{E}) = k/p \leq q/y \leq q^{3/4+\varepsilon}$. This concludes the proof in the case $g = 1$.

For $g = 2$, Proposition 5.4 in Section 5 of Chapter X of [15] implies that the cardinalities of elliptic curves $\mathcal{E}$ over $\mathbb{F}_q$ with $j$-invariant $j(\mathcal{E}) = 0, 1728$ take $O(1)$ values. Therefore we can choose $k$ and an elliptic curve $\mathcal{E}$ over $\mathbb{F}_q$ of exponent $\ell_q(\mathcal{E}) \leq q^{3/4+\varepsilon}$ as in the above with the additional condition $j(\mathcal{E}) \neq 0, 1728$. By Corollary 6 of [6] we see that there is a curve $\mathcal{C}$ of genus $g = 2$ such that the Jacobian $J_{\mathcal{C}}(\mathbb{F}_q)$ is isogenous to $\mathcal{E}(\mathbb{F}_q)\times\mathcal{E}(\mathbb{F}_q)$. Moreover, there exists an isogeny from $\mathcal{E}(\mathbb{F}_q)\times\mathcal{E}(\mathbb{F}_q)$ to $J_{\mathcal{C}}(\mathbb{F}_q)$, whose kernel (over an algebraic closure of $\mathbb{F}_q$) is isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. So $\ell_q(\mathcal{C}) \geq \ell_q(\mathcal{E})/2$, which concludes the proof for $g = 2$. $\square$

# 5 Proof of Theorem 3

The desired bound follows immediately from Theorems 1 and 6 of [5], that is, from (6) and (7), and the congruence $q - 1 \equiv 0 \pmod{m_g}$, where $m_i$, $i = 1, \ldots, 2g$, are as in (2). Again without loss of generality assume that $\varepsilon(x) \geq (\log x)^{-1/2}$. For $\eta = 2\varepsilon(x/2)$, similarly to (16), we see that the set $\mathcal{R}$ of primes $q \leq x$ such that $q-1$ has a divisor $m \in [x^{1/2-2\eta}, x^{1/2+2\eta}]$, is of cardinality $\#\mathcal{R} = o(x/\log x)$. Consider a prime $q \in (2x^{1-\eta}, x]$ which does not lie in $\mathcal{R}$, and any curve $\mathcal{C}$ of genus $g$ over $\mathbb{F}_q$. If $m_g > x^{1/2+\eta}$ then

$$\ell_q(\mathcal{C}) = m_{2g} \geq m_g > q^{1/2+\varepsilon(q)}.$$

Otherwise, by (3), $m_g \leq x^{1/2-2\eta}$ and by (1) we obtain

$$\begin{aligned} \ell_q(\mathcal{C}) &\geq \left(\frac{\#\mathcal{J}_{\mathcal{C}}(\mathbb{F}_q)}{m_1\cdots m_g}\right)^{1/g} \geq \left(\frac{(q^{1/2}-1)^{2g}}{m_1\cdots m_g}\right)^{1/g} \\ &\geq \left(\frac{x^{g-g\eta}}{x^{g/2-2g\eta}}\right)^{1/g} \geq x^{1/2+\eta} > q^{1/2+\varepsilon(q)} \end{aligned}$$

for large $x$. $\square$

## 6 Remarks

It is interesting to note that using (12) for other values of $s$ (besides $s = g$ and $s = 2g - 1$ as in the proof of Theorem 1) and thus corresponding sets $\mathcal{K}$, does not lead to any improvements.

**Open Question.** *Is the exponent in Theorem 1 sharp for arbitrary $g \geq 3$, as it is for $g = 1, 2$?*

Unfortunately the lack of knowledge about the distribution of possible cardinalities of Jacobians of curves of genus $g \geq 2$ prevents are from deriving an analogue of Theorem 2 for $g \geq 2$.

## References


[1] R. Avanzi, H. Cohen, C. Doche, G. Frey, T. Lange, K. Nguyen and F. Vercauteren, *Elliptic and hyperelliptic curve cryptography: Theory and practice*, CRC Press, 2005.

[2] H. Davenport, *Multiplicative number theory*, 2nd ed., Springer-Verlag, New York 1980.

[3] W. Duke, 'Almost all reductions modulo $p$ of an elliptic curve have a large exponent', *C. R. Math. Acad. Sci. Paris*, **337** (2003), 689–692.

[4] P. Erdős and R. Murty, 'On the order of $a \pmod p$', *Proc. 5th Canadian Number Theory Association Conf.* (R. Gupta and K. S. Williams, eds), Amer. Math. Soc., Providence, RI, 1999, 87–97.

[5] K. Ford, 'The distribution of integers with a divisor in a given interval', *Ann. Math.* **168** (2008), 367–433.

[6] E. W. Howe, F. Leprévost and B. Poonen, 'Large torsion subgroups of split Jacobians of curves of genus two or three', *Forum Math.*, **12** (2000), 315–364.

[7] K.-H. Indlekofer and N. M. Timofeev, 'Divisors of shifted primes', *Publ. Math. Debrecen*, **60** (2002), 307–345.

[8] F. Luca, J. McKee and I. E. Shparlinski, 'Small exponent point groups on elliptic curves', *J. Théorie des Nombres Bordeaux*, **18** (2006), 471–476.

[9] F. Luca and I. E. Shparlinski, 'On the exponent of the group of points on elliptic curves in extension fields', *Intern. Math. Research Notices*, **2005** (2005), 1391–1409.

[10] F. Pappalardi, 'On the order of finitely generated subgroups of $\mathbb{Q}^*$ (mod $p$) and divisors of $p-1$', *J. Number Theory*, **57** (1996), 207–222.

[11] K. Prachar, *Primzahlverteilung*, Springer, Berlin, 1957.

[12] H.-G. Rück, 'A note on elliptic curves over finite fields', *Math. Comp.*, **49** (1987), 301–304.

[13] R. Schoof, 'The exponents of the group of points on the reduction of an elliptic curve', *Arithmetic Algebraic Geometry* (G. van der Geer, F. Oort and J. Steenbrink, eds), Progr. Math., vol. 89, Birkhäuser, Boston, MA, 1991, 325–335.

[14] I. E. Shparlinski, 'Orders of points on elliptic curves', *Affine Algebraic Geometry* (J. Gutierrez, V. Shpilrain and J.-T. Yu, eds), Amer. Math. Soc., 2005, 245–252.

[15] J. H. Silverman, *The arithmetic of elliptic curves*, Springer-Verlag, Berlin, 1995.

[16] M. A. Tsfasman and S. G. Vlăduţ, *Algebraic-Geometric Codes*, Kluwer Acad. Pres, Dordrecht, 1991.

[17] J. F. Voloch, 'A note on elliptic curves over finite fields', *Bull. Soc. Math. France*, **116** (1988), 455–458.